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1976 J. Phys. A: Math. Gen. 9 1159

(http://iopscience.iop.org/0305-4470/9/7/018)

J. Phys. A: Math. Gen., Vol. 9, No. 7, 1976 Printed in Great Britain. © 1976

Initial value problem and causality of radiating oscillator

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Received 4 April 1975, in final form 23 February 1976

Abstract. The exact solution for the non-relativistic harmonic oscillator interacting with an electromagnetic field in the dipole approximation is derived with the help of the Heisenberg equation of motion. The space–time distribution of the field is reconstructed. The initial value problem for the field and for the motion of the oscillator is solved. The solutions obey macroscopic and microscopic causality conditions.

1. Introduction

The mutual interaction of fields and charges remains a basic problem in both classical and quantum electrodynamics. Due to the weakness of the electromagnetic coupling, most practical problems fall into one of two approximate schemes: either one looks for the motion of charges in a given electromagnetic field, or one looks for the electromagnetic field produced by given distributions of charges and currents.

An important historical example of radiating systems was an harmonically oscillating point current known as the Hertz dipole. Solving for the field produced by the dipole, Hertz treated the current as being given without considering the feedback of the radiation. There were attempts to account for the feedback in terms of modifications of the equation of motion for the charge. Instead of the real interaction with the self-field, the force of radiation friction was introduced. This approach when applied to the Hertz dipole gave a finite lifetime of excitation and the well known Lorentzian shape of the spectrum of emitted radiation.

This old problem, that of a radiating oscillator, can be exactly solved in the non-relativistic approximation even in the quantum case. The problem was solved by van Kampen (1951), who diagonalized the Hamiltonian of the model. Recently, a similar approach to the model was used by Eganova and Shirokov (1969) and Shirokov (1975). Norton and Watson (1959) discussed the ghost state problem in quantum field theory with the help of this model. They solved the boundary value problem in the momentum space for the field. The initial value problem for the field and oscillator variables was considered by Aichelburg (1966).

In our paper we solve directly the initial value problem for the Heisenberg equations of motion for the field and the variables describing the source (§ 2). The Heisenberg equations of motion are equivalent to the coupled Maxwell equations for the field together with Newton equations for the oscillating point charge.

We use our solution for the space–time reconstruction of the field. We demonstrate that our solutions strictly fulfil the causality condition. The proof requires a suitable choice of variables describing the system (§ 3).

This paper is a continuation of our previous paper (Żakowicz and Rzążewski 1974) where a similar system was solved with the rotating wave approximation. We point out that the non-exponential damping of the oscillator excitation found in this paper is absent when the counter-rotating terms are taken into account. This problem is discussed in detail by Shirokov (1975).

We complete our paper with a short discussion of non-equal time commutators for interacting fields in the system (§ 4).

2. The model and its solution

The model under consideration consists of a non-relativistic, spinless particle with charge e subject to an harmonic potential. A possible source of this potential is a uniformly charged medium spread out in a spherical region of radius large compared with the amplitude of oscillations. The total charge of this sphere is $-e$ to make the total system neutral. It is obviously reminiscent of the old Thomson model of the atom.

Our charged oscillator is interacting with the electromagnetic field through the minimal coupling. The Hamiltonian for the system reads:

$$H=\frac{1}{2m}(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2+\frac{1}{2}m\omega_0^2\boldsymbol{x}^2+\frac{1}{8\pi}\int \mathrm{d}_3\boldsymbol{r}(\boldsymbol{E}_\mathrm{T}^2+\boldsymbol{B}^2) \tag{2.1}$$

where m is the mass of the oscillating particle, ω_0 the frequency of its free oscillations, $\boldsymbol{x}$ the actual position of the particle, $\boldsymbol{p}$ its cannonical momentum, $\boldsymbol{A}$ the vector potential of the electromagnetic field, $\boldsymbol{E}_\mathrm{T}$ the transverse part of the electric field and $\boldsymbol{B}$ the magnetic field. Throughout this paper we use the radiation gauge, i.e., $\operatorname{div} \boldsymbol{A}=0$. The final results, however, will be presented in gauge-independent form. As usual in the radiation gauge the longitudinal part of the electric field is eliminated by direct Coulomb interaction, which constitutes an elastic potential in our case. We use a system of units in which $\hbar=1$, $c=1$.

We use the plane-wave decomposition of the transverse electromagnetic field, introducing the creation and annihilation operators $a^\dagger_{\boldsymbol{k}\mu}$ and $a_{\boldsymbol{k}\mu}$ of photons with definite wavevector $\boldsymbol{k}$ and linear polarization $\boldsymbol{e}_{\boldsymbol{k}\mu}$ ($\mu=1,2$). The vector potential has the following plane-wave decomposition:

$$\boldsymbol{A}(\boldsymbol{r})=\frac{1}{2\pi}\sum_\mu\int \mathrm{d}_3k\frac{1}{\sqrt{k}}\boldsymbol{e}_{\boldsymbol{k}\mu}(a_{\boldsymbol{k}\mu}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}}+a^\dagger_{\boldsymbol{k}\mu}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{r}}). \tag{2.2}$$

The creation and annihilation operators $a^\dagger_{\boldsymbol{u}\mu}$ and $a_{\boldsymbol{u}\mu}$ satisfy commutation relations of the following form:

$$[a_{\boldsymbol{k}\mu}, a^\dagger_{\boldsymbol{k}'\mu'}]=\delta_{\mu\mu'}\delta_3(\boldsymbol{k}-\boldsymbol{k}') \qquad [a_{\boldsymbol{k}\mu}, a_{\boldsymbol{k}'\mu'}]=0=[a^\dagger_{\boldsymbol{k}\mu}, a_{\boldsymbol{k}'\mu'}]. \tag{2.3}$$

The only simplifying assumption will be the dipole approximation, which means that we assume the amplitude of oscillation to be small compared to the wavelength of the resonant radiation. Technically, this means putting $\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}}=1$ in the Hamiltonian (2.1).

At this point we note that such a model would be ultraviolet divergent. This divergence has no basic physical meaning and is related partially to the incorrect treatment of large wavevectors in the dipole approximation. As the theory with dipole coupling cannot have a universal character, we do not hope that one can remove this divergence by a satisfactory renormalization procedure. Therefore, it is more appropriate to introduce a form factor tempering the coupling for high frequencies just as the term $e^{ik.r}$ does. A very convenient form of the form factor $g(k)$ replacing $k^{-1/2}$ in the decomposition of the vector potential in the Hamiltonian is:

$$g(k)=\frac{1}{\sqrt{k}}\frac{\beta}{(\beta^2+k^2)^{1/2}} \tag{2.4}$$

where the cut-off parameter β is much larger than the resonant frequency and is of the order $\beta \sim d^{-1}$ where d is equal to the amplitude of oscillations. In the preceding paper (Żakowicz and Rzążewski 1974) we have shown how to reduce the problem of N harmonic oscillators contained in a small volume to the Hamiltonian of type (2.1). If that is the case, the coupling constant e contains an additional factor $\sqrt{N}$ considerably increasing the coupling of the system with the radiation. It also makes the $\boldsymbol{A}^2$ term more and more relevant.

We will solve the model by finding the time evolution of the operators in the Heisenberg picture. The equations of motion of the operators $\boldsymbol{x}$, $\boldsymbol{p}$, $a_{k\mu}$, $a^\dagger_{k\mu}$ are

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t=-m\omega_0^2\boldsymbol{x} \tag{2.5a}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\frac{\boldsymbol{p}}{m}-\frac{e}{2\pi m}\sum_\mu\int \mathrm{d}_3k\boldsymbol{e}_{k\mu}g(k)(a_{k\mu}+a^\dagger_{k\mu}) \tag{2.5b}$$

$$\frac{\mathrm{d}a^\dagger_{k\mu}}{\mathrm{d}t}=\mathrm{i}ka^\dagger_{k\mu}-\mathrm{i}\frac{e}{2\pi m}g(k)\boldsymbol{e}_{k\mu}\cdot\boldsymbol{p}+\mathrm{i}\frac{e^2}{4\pi^2m}g(k)\boldsymbol{e}_{k\mu}\sum_\nu\int \mathrm{d}_3p\boldsymbol{e}_{p\nu}g(p)(a_{p\nu}+a^\dagger_{p\nu}) \tag{2.5c}$$

$$\frac{\mathrm{d}a_{k\mu}}{\mathrm{d}t}=-\mathrm{i}ka_{k\mu}+\mathrm{i}\frac{e}{2\pi m}g(k)\boldsymbol{e}_{k\mu}\cdot\boldsymbol{p}-\mathrm{i}\frac{e^2}{4\pi^2m}g(k)\boldsymbol{e}_{k\mu}\sum_\nu\int \mathrm{d}_3p\boldsymbol{e}_{p\nu}g(p)(a_{p\nu}+a^\dagger_{p\nu}). \tag{2.5d}$$

As in Żakowicz and Rzążewski (1974) this set of linear integro-differential equations can be conveniently solved with the help of the Laplace transform in the time variable. The integral part of equations (2.5) causes no problem because it is of separable form.

The solutions expressed by the inverse Laplace transform read:

$$\begin{aligned}\boldsymbol{x}(t)=&\left(1-\omega_0^2\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{zH(z)}\right)\boldsymbol{x}(0)+\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)}\frac{\boldsymbol{p}(0)}{m}\\&-\frac{e}{2\pi m}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z\mathrm{e}^{zt}}{H(z)}\sum_\mu\int \mathrm{d}_3kg(k)\boldsymbol{e}_{k\mu}\left(\frac{a_{k\mu}(0)}{z+\mathrm{i}k}+\frac{a^\dagger_{k\mu}(0)}{z-\mathrm{i}k}\right)\end{aligned} \tag{2.6a}$$

$$\begin{aligned}\boldsymbol{p}(t)=&\,m\omega_0^4\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{z^2H(z)}\boldsymbol{x}(0)+\left(1-\omega_0^2\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{zH(z)}\right)\boldsymbol{p}(0)\\&+\frac{e\omega_0^2}{2\pi}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)}\sum_\mu\int \mathrm{d}_3kg(k)\boldsymbol{e}_{k\mu}\left(\frac{a_{k\mu}(0)}{z+\mathrm{i}k}+\frac{a^\dagger_{k\mu}(0)}{z-\mathrm{i}k}\right)\end{aligned} \tag{2.6b}$$

$$\begin{aligned}a_{k\mu}(t)=&\,a_{k\mu}(0)\,\mathrm{e}^{-\mathrm{i}kt}+\mathrm{i}\frac{e}{2\pi}g(k)\boldsymbol{e}_{k\mu}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{(z+\mathrm{i}k)H(z)}\left(z\frac{\boldsymbol{p}(0)}{m}-\omega_0^2\boldsymbol{x}(0)\right)\\&-\mathrm{i}\frac{e^2}{4\pi^2m}g(k)\boldsymbol{e}_{k\mu}\int\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z^2\mathrm{e}^{zt}}{H(z)(z+\mathrm{i}k)}\sum_\nu\int \mathrm{d}_3pg(p)\boldsymbol{e}_{p\nu}\left(\frac{a_{p\nu}(0)}{z+\mathrm{i}p}+\frac{a^\dagger_{p\nu}(0)}{z-\mathrm{i}p}\right).\end{aligned} \tag{2.6c}$$

The solution for the operator $a^\dagger_{k\mu}$ can be obtained from $a_{k\mu}$ by Hermitian conjugation.

It is easy to check that one does not need to conjugate z, provided the contour Γ remains the same as well as the measure $dz/2\pi i$.

According to the theory of the Laplace transform, (2.6) are the solutions for positive t if the contour Γ lies parallel to the imaginary axis in the complex z plane to the right of all singularities of the integrands.

To describe the contour Γ in detail we must know the function $H(z)$ entering most terms in (2.6):

$$H(z)=\omega_0^2+z^2\left(1+\frac{4e^2}{3\pi m}\int_0^\infty dk\frac{k^3g^2(k)}{z^2+k^2}\right). \tag{2.7}$$

The integrand is an even function of k and we can extend the integration over k from $-\infty$ to $+\infty$.

From this it is clear that, due to the form of the denominator in the integrand, $H(z)$ is double-valued. The cut extends along the whole imaginary axis.

It is interesting to compare the properties of $H(z)$ with its counterpart $h(z)$ of Żakowicz and Rzążewski (1974). The function $h(z)$ was multi-valued having a logarithmic branch point at $z=0$. Retaining counter-rotating terms in the Hamiltonian restores the basic symmetry between positive and negative frequencies, known as 'crossing symmetry', coming from all relativistic field theories.

We shall examine in more detail the two branches of $H(z)$. If we calculate the integral in (2.7) assuming $\operatorname{Re} z>0$ we get the branch having no zeros to the right of the imaginary axis. Assuming $\operatorname{Re} z<0$ when calculating $H(z)$, one obtains the branch without zeros in the left half-plane. The first branch is appropriate for finding the evolution into the future, while the second serves for the evolution into the past. Being interested in the future evolution of the system we choose the contour Γ in the integrals in (2.6) to be to the right of the imaginary axis. The branch of $H(z)$ for the future can be easily computed. Due to our choice of the form factor it has the simple rational form.

$$H_+(z)=\omega_0^2+z^2\left(1+\frac{2e^2}{3m}\frac{\beta^2}{\beta+z}\right). \tag{2.8}$$

In the following we shall omit the subscript $(+)$ because only (2.8) will be used in the calculations.

The only type of singularity produced by inserting (2.8) into (2.6) is a pole.

The function $H(z)$ has exactly three zeros. They can be found explicitly by means of the well known Cardan formulae since they are roots of a cubic equation. This cubic equation has only one real root, the remaining two roots being mutually complex conjugate. Their approximate values are given by:

$$\begin{aligned} z_{1,2}&=\pm i\omega_0\frac{1}{(1+\frac{2}{3}e^2\beta/m)^{1/2}}-\frac{1}{3}\frac{e^2\omega_0^2}{m}\frac{1}{(1+\frac{2}{3}e^2\beta/m)^2}\\ z_3&=-\beta(1+\tfrac{2}{3}e^2\beta/m). \end{aligned} \tag{2.9}$$

These expressions are valid for an arbitrary value of the coupling constant e. They can be found noticing that for the complex roots $z_{1,2}$ we have $|\operatorname{Re} z_{1,2}|/|\operatorname{Im} z_{1,2}|\ll 1$ and this ratio can be treated as the expansion parameter†. For our contour Γ, integrals over z in the solutions (2.6) can be calculated with the help of the theorem of residues. To this end we must close the contour by a semicircle lying in the left half-plane. Integrals over this

† This procedure was suggested by Professor P O Fröman.

semicircle do not contribute and therefore each integral in (2.6) is a sum of a few terms coming from the poles and having purely exponential dependence on the time variable.

Physical processes of damping, emission and scattering are related to zeros z_1 and z_2. Phenomena related to z_3 remain beyond experimental possibilities due to their very short time scale, $\tau \sim \beta^{-1}$, compared with the single oscillation period, $T \simeq \omega_0^{-1}$. Other choices of the form factor could drastically change the analytic properties of solutions. Instead of zero z_3 we could get several zeros as well as some branch points. However, this would modify the solutions only for that very short time.

It is interesting to investigate the behaviour of zeros $z_{1,2}$ when the coupling constant e is varying. Increasing its value from 0 we get continuous decreasing of $|\mathrm{Im}\ z_{1,2}|$ going finally to 0. The value of $|\mathrm{Re}\ z_{1,2}|$ is at first increasing but then it starts to decrease and tends to 0 faster than $|\mathrm{Im}\ z_{1,2}|$. The position of zeros is indicated on figure 1. This result, that the lifetime of excitation can grow with growing coupling, seems to be beyond our experience based on the weak coupling limit and perturbative approaches.

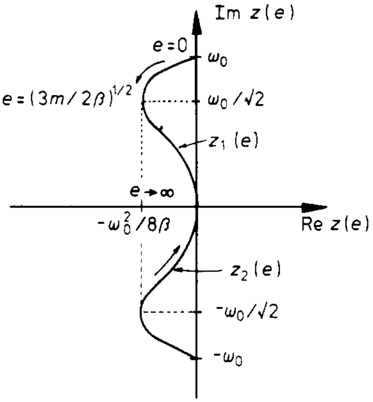

Figure 1. The position of the zeros of $H(z)$ as a function of the coupling constant e.

Again it is interesting to compare the present solutions with that given in Żakowicz and Rzążewski (1974). In that paper there were not only contributions from poles but also from the cut. The contribution from the cut gave rise to the non-exponential tail of the evolution and led to the violation of relativistic causality. Leaving the discussion of causality in the present model until § 3, we will now comment very briefly on the damping.

Suppose we start the evolution from the state $|\psi_0\rangle$ of an excited oscillator and photon vacuum. The basic quantity describing the damping in such a state, which can be computed easily with the help of our solutions, is the expectation value of the oscillator excitation. To define such a quantity, which tends to 0 when $t \to \infty$, we must subtract the quantum fluctuation of energy which appears for the oscillator when it is in the vacuum state $|\Omega_{\mathrm{os}}\rangle$:

$$\mathscr{E}_{\mathrm{os}}(t) = \langle \psi_0 H_0(t) \psi_0 \rangle - \langle \Omega_{\mathrm{os}} H_0(t) \Omega_{\mathrm{os}} \rangle \tag{2.10}$$

where

$$H_0(t) = \frac{1}{2m} \boldsymbol{p}^2(t) + \frac{1}{2} m \omega_0^2 \boldsymbol{x}^2(t).$$

According to our discussion this function is a combination of damped exponential functions of time multiplied by trigonometric functions of time. This fact was pointed out recently by Shirokov (1975).

3. Space-time structure of the solutions

The standard perturbative approach to the quantum radiation problem does not lead directly to the description of the space–time structure of the electromagnetic field. Knowledge of approximate transition probabilities is usually sufficient for directional and spectral properties of radiation. It is too crude, however, to pass to the space–time picture. The quantum discussion of the emission process is typically given in terms of photons, and because the photon is not a localizable object it causes additional difficulties in passing to the space–time picture of radiation.

Having known the exact time dependence of the field and source variables $a^{\dagger}_{k\mu}(t)$, $a_{k\mu}(t)$, $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$, we are able to perform the complete reconstruction of the electromagnetic field in space–time.

In the Hamiltonian (2.1) we needed the vector potential of the field only at the point occupied by charge. Choosing the right coordinate system, in the dipole approximation we set $e^{i\boldsymbol{k}.\boldsymbol{r}}=1$. Now, to find the field in the whole space we have to keep this exponential factor in the formula (2.2), substituting $a_{k\mu}(t)$, $a^{\dagger}_{k\mu}(t)$ given by (2.6*c*).

Besides the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ contributing to the transverse part of the field we also need the scalar potential $\phi(\boldsymbol{r}, t)$, giving the longitudinal part of the electric field. Inside the system this scalar potential had the form $\phi(\boldsymbol{x}, t)=\frac{1}{2}e^{-1}m\omega_0^2\boldsymbol{x}^2$. Outside the system the scalar potential is that of the electric dipole:

$$\phi(\boldsymbol{r}, t)=e\boldsymbol{x}\,.\,\boldsymbol{r}/r^3. \tag{3.1}$$

The total electric field outside the source, expressed by $a_{k\mu}(t)$, $a^{\dagger}_{k\mu}(t)$, $\boldsymbol{x}(t)$, can be found from the relation:

$$\boldsymbol{E}(\boldsymbol{r}, t)=-\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r}, t)-\nabla\phi(\boldsymbol{r}, t). \tag{3.2}$$

It is equal to:

$$\boldsymbol{E}(\boldsymbol{r}, t)=\frac{i}{2\pi}\sum_{\mu}\int d_3k\,\boldsymbol{e}_{k\mu}\sqrt{k}\,(a_{k\mu}(t)\,e^{i\boldsymbol{k}.\boldsymbol{r}}-a^{\dagger}_{k\mu}(t)\,e^{-i\boldsymbol{k}.\boldsymbol{r}})-\frac{e}{r^3}(\hat{I}-3\boldsymbol{nn}).\,\boldsymbol{x}(t) \tag{3.3}$$

where $\boldsymbol{n}=\boldsymbol{r}/r$ and $\boldsymbol{nn}$ denotes its dyadic product.

The magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)=\operatorname{curl}\boldsymbol{A}(\boldsymbol{r}, t)$ is equal to:

$$\boldsymbol{B}(\boldsymbol{r}, t)=\frac{i}{2\pi}\sum_{\mu}\int d_3k\frac{\boldsymbol{k}\times\boldsymbol{e}_{k\mu}}{\sqrt{k}}(a_{k\mu}(t)\,e^{i\boldsymbol{k}.\boldsymbol{r}}-a^{\dagger}_{k\mu}(t)\,e^{-i\boldsymbol{k}.\boldsymbol{r}}). \tag{3.4}$$

Now we insert into the formulae (3.3) and (3.4) the expressions for $a_{k\mu}(t)$, $a^{\dagger}_{k\mu}(t)$, $\boldsymbol{x}(t)$ given by (2.6), and express $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ in terms of the initial operators $a_{k\mu}(0)$, $a^{\dagger}_{k\mu}(0)$, $\boldsymbol{x}(0)$, $\boldsymbol{p}(0)$.

All integrations over wavevectors $\boldsymbol{k}$ can be effectively performed using spherical coordinates. In the appendix we give some integrals useful in these integrations.

It is useful to notice that though the integration over the radial variable k extends from 0 to ∞, in all cases the integrands are even and the integration can be extended from $-\infty$ to ∞. We can simplify these integrations if instead of the form factor $g(k)$, appearing manifestly in (2.6), we put again $k^{-1/2}$. The form factor was necessary to remove divergences introduced by the dipole approximation; however, outside the system these divergencies do not occur due to the presence of the oscillating factor $e^{i\boldsymbol{k}.\boldsymbol{r}}$. All regularizations introduced by the form factor $g(k)$ are represented in the function $H(z)$.

If in making the field reconstruction we had left the form factor in the integrand, we would have, in addition to poles in the complex k plane, also the branch points at $k=\pm i\beta$. Therefore, we would get an additional contribution from integrals along the cuts. This contribution would represent the extension of the source. However, the time taken by the signal in passing through the source is of the order $t_p=\beta^{-1}$ and therefore very short. Previously, discussing the damping of the source, we have disregarded such fast processes. We will neglect them also in the discussion of field propagation and causality. Taking advantage of the fact that the expressions discussed above are finite when $g(k)\to k^{-1/2}$ we pass to this limit. This leads to fields produced by a point-like source for which the discussion of causality is particularly simple.

After this discussion we write down the expressions for the electric, $\boldsymbol{E}(\boldsymbol{r},t)$, and magnetic, $\boldsymbol{B}(\boldsymbol{r},t)$, fields:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t)=&\frac{i}{2\pi}\sum_\mu\int d_3k\boldsymbol{e}_{\boldsymbol{k}\mu}\sqrt{k}(a_{\boldsymbol{k}\mu}(0)\,e^{i(\boldsymbol{k}.\boldsymbol{r}-kt)}-a^\dagger_{\boldsymbol{k}\mu}(0)\,e^{-i(\boldsymbol{k}.\boldsymbol{r}-kt)}\\&-e\int_\Gamma\frac{dz}{2\pi i}\frac{e^{z(t-r)}}{H(z)}\left[\frac{z}{r}(\hat{I}-\boldsymbol{nn})+\left(\frac{1}{r^2}+\frac{1}{zr^3}\right)(\hat{I}-3\boldsymbol{nn})\right]\left(z\frac{\boldsymbol{p}(0)}{m}-\omega_0^2\boldsymbol{x}(0)\right)\\&-e\frac{1}{r^3}(\hat{I}-3\boldsymbol{nn}).\boldsymbol{x}(0)\\&+\frac{e^2}{2\pi m}\int_\Gamma\frac{dz}{2\pi i}\frac{z^2e^{z(t-r)}}{H(z)}\left[\frac{z}{r}(\hat{I}-\boldsymbol{nn})+\left(\frac{1}{r^2}+\frac{1}{zr^3}\right)(\hat{I}-3\boldsymbol{nn})\right]\\&\times\sum_\nu\int d_3p\frac{\boldsymbol{e}_{\boldsymbol{p}\nu}}{\sqrt{p}}\left(\frac{a_{\boldsymbol{p}\nu}(0)}{z+ip}+\frac{a^\dagger_{\boldsymbol{p}\nu}(0)}{z-ip}\right)\end{aligned}\qquad(3.5)$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r},t)=&\frac{i}{2\pi}\sum_\mu\int d_3k\frac{\boldsymbol{k}\times\boldsymbol{e}_{\boldsymbol{k}\mu}}{\sqrt{k}}(a_{\boldsymbol{k}\mu}(0)\,e^{i(\boldsymbol{k}.\boldsymbol{r}-kt)}-a^\dagger_{\boldsymbol{k}\mu}(0)\,e^{-i(\boldsymbol{k}.\boldsymbol{r}-kt)})\\&-e\int\frac{dz}{2\pi i}\frac{e^{z(t-r)}}{H(z)}\left(\frac{z}{r}+\frac{1}{r^2}\right)\left(z\boldsymbol{n}\times\frac{\boldsymbol{p}(0)}{m}-\omega_0^2\boldsymbol{n}\times\boldsymbol{x}(0)\right)\\&+\frac{e^2}{2\pi m}\int\frac{dz}{2\pi i}\frac{z^2e^{z(t-r)}}{H(z)}\left(\frac{z}{r}+\frac{1}{r^2}\right)\sum_\nu\int d_3p\frac{\boldsymbol{n}\times\boldsymbol{e}_{\boldsymbol{p}\nu}}{\sqrt{p}}\left(\frac{a_{\boldsymbol{p}\nu}(0)}{z+ip}+\frac{a^\dagger_{\boldsymbol{p}\nu}(0)}{z-ip}\right).\end{aligned}\qquad(3.6)$$

Now we will discuss the structure of these formulae. The first terms in (3.5) and (3.6) represent contributions to the field coming from the free propagation. The other terms represent the field radiated by the source and the scattered field. The distinction between the source part and the scattered part is important in a discussion of causal properties of the evolution of the system, so we comment on this point in more detail. At first sight this distinction seems to be evident. However, there is a certain trap in it, which we now illustrate.

At first one would think that the terms proportional to the oscillator position and momentum operators $\boldsymbol{x}(0)$ and $\boldsymbol{p}(0)$ contribute to the source part, and the rest, i.e. the last terms proportional to the initial field variables, contribute to the scattered part. This opinion is supported by the fact that such a 'source' field exhibits the ideal causal properties characteristic of emission. However, this identification is wrong. To show this we have to investigate how fields initially present, $\boldsymbol{E}_T(\boldsymbol{r}, 0)$ and $\boldsymbol{B}(\boldsymbol{r}, 0)$, influence the field at later times, as well as how they influence the excitation of the oscillator. The latter can be checked by direct inspection of $\boldsymbol{x}(t), \boldsymbol{p}(t)$. If our identification were correct, the initial field at the point $\boldsymbol{r}'$ should not influence the scattered field at the point $\boldsymbol{r}$ for time $t < r + r'$, i.e. the time needed to reach the source at $\boldsymbol{x} = 0$ and then to come to the point $\boldsymbol{r}$.

To investigate this we introduce initial field operators $\boldsymbol{E}_T(\boldsymbol{r}, 0)$, $\boldsymbol{B}(\boldsymbol{r}, 0)$ instead of $a_{\boldsymbol{p}\nu}(0)$, $a^\dagger_{\boldsymbol{p}\nu}(0)$. From the expressions (3.3) and (3.4) one can easily find that

$$\sum_\nu \boldsymbol{e}_{\boldsymbol{p}\nu} a_{\boldsymbol{p}\nu}(0) = -\mathrm{i}\frac{\pi}{\sqrt{p}} \int \frac{\mathrm{d}_3 r'}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}'} \left(\boldsymbol{E}_T(\boldsymbol{r}', 0) - \frac{\boldsymbol{p}}{p} \times \boldsymbol{B}(\boldsymbol{r}', 0) \right) \tag{3.7a}$$

$$\sum_\nu \boldsymbol{e}_{\boldsymbol{p}\nu} a^\dagger_{\boldsymbol{p}\nu}(0) = \mathrm{i}\frac{\pi}{\sqrt{p}} \int \frac{\mathrm{d}_3 r'}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{p}.\boldsymbol{r}'} \left(\boldsymbol{E}_T(\boldsymbol{r}', 0) - \frac{\boldsymbol{p}}{p} \times \boldsymbol{B}(\boldsymbol{r}', 0) \right). \tag{3.7b}$$

The 'scattered' part of the field and the operators of the oscillator contain the field-dependent term

$$C = \sum_\nu \int \mathrm{d}_3 p \frac{\boldsymbol{e}_{\boldsymbol{p}\nu}}{\sqrt{p}} \left(\frac{a_{\boldsymbol{p}\nu}(0)}{z + \mathrm{i}p} + \frac{a^\dagger_{\boldsymbol{p}\nu}(0)}{z - \mathrm{i}p} \right). \tag{3.8}$$

Putting (3.7) in (3.8) one gets

$$C = \frac{1}{2} \int \mathrm{d}_3 r' \left[\left(\boldsymbol{E}_T(\boldsymbol{r}', 0) - \frac{1}{z} \mathrm{curl}\, \boldsymbol{B}(\boldsymbol{r}', 0) \right) \frac{\mathrm{e}^{-zr'}}{r'} + \frac{1}{zr'} \mathrm{curl}\, \boldsymbol{B}(\boldsymbol{r}', 0) \right]. \tag{3.9}$$

We see that the last term, related to the initial distribution of magnetic field operator $\boldsymbol{B}(\boldsymbol{r}', 0)$ would influence instantaneously the change of $\boldsymbol{x}(t), \boldsymbol{p}(t)$ and after $t > r$ also the field at the point $\boldsymbol{r}$.

The first term in (3.9), and, what is interesting, the contribution coming from the initial electric field, satisfy our condition; introduced into the 'scattered' part of $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ it gives no contribution when $t < r + r'$.

Why does the initial magnetic field cause the trouble?

The answer to this question is very instructive. Our troubles are consequences of incorrect separation of the radiation into source and scattered part. The canonical momentum $\boldsymbol{p}$ also contains the field part $e\boldsymbol{A}$:

$$\boldsymbol{p} = \boldsymbol{\pi} + e\boldsymbol{A} \tag{3.10}$$

where $\boldsymbol{\pi} = m\dot{\boldsymbol{x}}$ is the kinematic momentum.

The proper source variables are the position $\boldsymbol{x}$ of the charge and its kinematic momentum $\boldsymbol{\pi}$. Taking these variables as a basis for our separation into source and scattered parts, we will show that radiative processes satisfy all the requirements of causality. They have the additional advantage that they do not depend on the gauge and will allow us to express the solution of our problem in a completely gauge-independent form. Using these variables we will write the solution of our model equivalent to (2.6):

$$\boldsymbol{x}(t)=\boldsymbol{x}(0)\left(1-\omega_0^2\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{zH(z)}\right)+\frac{\boldsymbol{\pi}(0)}{m}\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)}$$

$$+\mathrm{i}\frac{e}{2\pi m}\int \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)}\sum_\nu\int \mathrm{d}_3p\sqrt{p}\boldsymbol{e}_{p\nu}\left(\frac{a_{p\nu}(0)}{z+\mathrm{i}p}-\frac{a^\dagger_{p\nu}(0)}{z-\mathrm{i}p}\right) \tag{3.11a}$$

$$\boldsymbol{\pi}(t)=m\dot{\boldsymbol{x}}(t) \tag{3.11b}$$

$$a^\dagger_{k\mu}(t)=\mathrm{e}^{\mathrm{i}kt}a^\dagger_{k\mu}(0)+\mathrm{i}\frac{e}{2\pi}g(k)\boldsymbol{e}_{k\mu}\int\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)(z-\mathrm{i}k)}\left(\omega_0^2\boldsymbol{x}(0)-z\frac{\boldsymbol{\pi}(0)}{m}\right)$$

$$+\frac{e^2}{4\pi^2 m}g(k)\boldsymbol{e}_{k\mu}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z\,\mathrm{e}^{zt}}{H(z)(z-\mathrm{i}k)}\sum_\nu\int\mathrm{d}_3p\sqrt{p}\boldsymbol{e}_{p\gamma}\left(\frac{a_{p\nu}(0)}{z+\mathrm{i}p}-\frac{a^\dagger_{p\nu}(0)}{z-\mathrm{i}p}\right). \tag{3.11c}$$

The annihilation operator $a_{k\mu}(t)$ can be found from (3.11c) by conjugation.

These solutions will now be used to relate the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ at time t with initial data. The initial data consist of the state of motion for oscillator $\boldsymbol{x}(0)$ and $\boldsymbol{\pi}(0)$, magnetic field $\boldsymbol{B}(\boldsymbol{r},0)$ and transverse part of the electric field $\boldsymbol{E}_\mathrm{T}(\boldsymbol{r},0)$. The longitudinal part of the electric field is uniquely determined by the state of the source.

Using (3.7) one can easily find that

$$\sum_\nu\int\mathrm{d}_3p\sqrt{p}\boldsymbol{e}_{p\nu}\left(\frac{a_{p\nu}(0)}{z+\mathrm{i}p}-\frac{a^\dagger_{p\nu}(0)}{z-\mathrm{i}p}\right)=-\frac{\mathrm{i}}{2}\int\mathrm{d}_3r'\frac{\mathrm{e}^{-zr'}}{r'}(z\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)+\operatorname{curl}\boldsymbol{B}(\boldsymbol{r}',0)). \tag{3.12}$$

According to our discussion we see how the electric and magnetic fields can be separated into free (fr), source (so), and scattered (sc) parts:

$$\boldsymbol{E}=\boldsymbol{E}^{\mathrm{fr}}+\boldsymbol{E}^{\mathrm{so}}+\boldsymbol{E}^{\mathrm{sc}} \tag{3.13a}$$

$$\boldsymbol{B}=\boldsymbol{B}^{\mathrm{fr}}+\boldsymbol{B}^{\mathrm{so}}+\boldsymbol{B}^{\mathrm{sc}}. \tag{3.13b}$$

As the formulae are rather long we will write each part separately:

$$\boldsymbol{E}^{\mathrm{fr}}(\boldsymbol{r},t)=\frac{1}{4\pi}\int\mathrm{d}_3r'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}(\delta'(t-|\boldsymbol{r}-\boldsymbol{r}'|)\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)+\delta(t-|\boldsymbol{r}-\boldsymbol{r}'|)\operatorname{curl}\boldsymbol{B}(\boldsymbol{r}',0)) \tag{3.14a}$$

$$\boldsymbol{E}^{\mathrm{so}}(\boldsymbol{r},t)=-e\frac{\hat{I}-3\boldsymbol{nn}}{r^3}\cdot\boldsymbol{x}(0)+e\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r)}}{H(z)}\left[\frac{z}{r}(\hat{I}-\boldsymbol{nn})+\left(\frac{1}{r^2}+\frac{1}{zr^3}\right)(\hat{I}-3\boldsymbol{nn})\right]$$

$$\cdot\left(\omega_0^2\boldsymbol{x}(0)-\frac{z\boldsymbol{\pi}(0)}{m}\right) \tag{3.14b}$$

$$\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r},t)=-\frac{e^2}{4\pi m}\int\mathrm{d}_3r'\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z\,\mathrm{e}^{z(t-r-r')}}{H(z)}\frac{1}{r'}\left[\frac{z}{r}(\hat{I}-\boldsymbol{nn})+\left(\frac{1}{r^2}+\frac{1}{zr^3}\right)(\hat{I}-3\boldsymbol{nn})\right]$$

$$\cdot(z\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)+\operatorname{curl}\boldsymbol{B}(\boldsymbol{r}',0)) \tag{3.14c}$$

$$\boldsymbol{B}^{\mathrm{fr}}(\boldsymbol{r},t)=\frac{1}{4\pi}\int\mathrm{d}_3r'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}(\delta'(t-|\boldsymbol{r}-\boldsymbol{r}'|)\boldsymbol{B}(\boldsymbol{r}',0)-\delta(t-|\boldsymbol{r}-\boldsymbol{r}'|)\operatorname{curl}\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)) \tag{3.15a}$$

$$\boldsymbol{B}^{\mathrm{so}}(\boldsymbol{r},t)=e\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r)}}{H(z)}\left(\frac{z}{r}+\frac{1}{r^2}\right)\boldsymbol{n}\times\left(\omega_0^2\boldsymbol{x}(0)-z\frac{\boldsymbol{\pi}(0)}{m}\right) \tag{3.15b}$$

$$\boldsymbol{B}^{\mathrm{sc}}(\boldsymbol{r},t)=-\frac{e^2}{4\pi m}\int \mathrm{d}_3r'\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z\mathrm{e}^{z(t-r-r')}}{H(z)}\frac{1}{r'}\Big(\frac{z}{r}+\frac{1}{r^2}\Big)\boldsymbol{n}\times(z\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)+\operatorname{curl}\boldsymbol{B}(\boldsymbol{r}',0)). \tag{3.15c}$$

We will complete these formulae by writing the motion of the charge:

$$\boldsymbol{x}(t)=\boldsymbol{x}(0)\Big(1-\omega_0^2\int\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{zH(z)}\Big)+\frac{\boldsymbol{\pi}(0)}{m}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{zt}}{H(z)}$$
$$+\frac{e}{4\pi m}\int\mathrm{d}_3r'\frac{1}{r'}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r')}}{H(z)}(z\boldsymbol{E}_\mathrm{T}(\boldsymbol{r}',0)+\operatorname{curl}\boldsymbol{B}(\boldsymbol{r}',0)). \tag{3.16}$$

According to (3.14b) the source part of the electric field $\boldsymbol{E}^{\mathrm{so}}(\boldsymbol{r},t)$ consists initially of a static longitudinal dipole field. The field at the point $\boldsymbol{r}$ is not modified for $t<r$. Before the moment $t=r$ the integral over the contour Γ can be closed by a semicircle lying in the right half-plane. In this region the function $H(z)$ has no zeros and therefore the integral vanishes. For $t>r$ one has to close the contour Γ in the left half-plane and due to zeros of $H(z)$ the integral is different from zero, representing the emitted radiation. Because the real parts of the zeros of $H(z)$ are negative, the amplitude of a passing wave at a given point is exponentially decreasing to zero when $t-r$ tends to infinity. It is interesting to notice that the term proportional to r^{-3} of the source part of the electric field has a pole also at $z=0$. The contribution of this pole for $t>r$ exactly cancels the static dipole field. Therefore, this static field exists only before the front of the outgoing wave. Behind the front there is only a freely propagating field. Finally, the source is damped to the ground state and $\lim_{t\to\infty}\boldsymbol{E}^{\mathrm{so}}(\boldsymbol{r},t)=0$.

The scattered part of electric field $\boldsymbol{E}^{\mathrm{so}}$ also has a clear interpretation. Due to a very similar mechanism, the initial field at the point $\boldsymbol{r}'$ can contribute to the field at the point $\boldsymbol{r}$ only when $t>r+r'$. This initial field can excite the oscillator only after $t>r'$ (compare (3.16)) and then the oscillator starts to radiate. This radiation can reach the point $\boldsymbol{r}$ after an additional time equal to r. We interpret this as a scattering process.

The interpretation of terms representing the magnetic field (3.15) is very similar.

The motion of the charge is described by (3.16). The first two terms describe its damped oscillations. The damping is due to radiative friction. In the standard approach, this friction is represented in equations of motion by the force $\boldsymbol{F}_\mathrm{f}=\frac{2}{3}e^2\dddot{\boldsymbol{x}}$. Our motion is different to that obtained from equations with $\boldsymbol{F}_\mathrm{f}$. In particular it does not show self-acceleration, this being an unphysical consequence of using $\boldsymbol{F}_\mathrm{f}$.

The last term in (3.16) represents the excitation of the oscillator by a field that is initially present. It follows from that expression that the initial field at the point $\boldsymbol{r}'$ can influence the motion of the oscillator only for $t>r'$.

All these expressions show manifestly that exact relativistic macroscopic causality is satisfied in processes involving the interaction of radiation with a harmonic oscillator.

4. Field commutators and microscopic causality

Until now our discussion of causality has been related to propagation of signals; sometimes called the macroscopic causality. Another way of expressing the causality condition is connected with properties of the field commutators at different space–time points. The field operators at two space-like separated points should commute which means that measurements of fields at these points should not interfere. This concept of causality is sometimes called the microscopic causality or locality.

Using our solutions we give now a short discussion of field commutators at different space–time points. Since the fields at time t are linear combinations of initial fields and source variables, the commutators are c numbers. The system is translationally invariant with respect to the time variable. Therefore, it is sufficient to consider the commutators with one field taken at the initial moment.

Starting from (3.5) and (3.6) one can find the following values of the commutators:

$$[E^i(\boldsymbol{r}, 0), E^j(\boldsymbol{r}', t)] = 4\pi\mathrm{i}\left(-\delta_{ij}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r^i}\frac{\partial}{\partial r^j}\right)\mathscr{D}_0(\boldsymbol{r}-\boldsymbol{r}', t)$$
$$+\mathrm{i}\frac{e^2}{m}\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r-r')}}{H(z)}\left[\frac{z^2}{r}(\delta_{is}-n_in_s)+\left(\frac{z}{r^2}+\frac{1}{r^3}\right)(\delta_{is}-3n_in_s)\right]$$
$$\times\left[\frac{z^2}{r'}(\delta_{js}-n'_jn'_s)+\left(\frac{z}{r'^2}+\frac{1}{r'^3}\right)(\delta_{js}-3n'_jn'_s)\right] \tag{4.1}$$

$$[B^i(\boldsymbol{r}, 0), B^j(\boldsymbol{r}', t)] = 4\pi\mathrm{i}\left(-\delta_{ij}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r^i}\frac{\partial}{\partial r^j}\right)\mathscr{D}_0(\boldsymbol{r}-\boldsymbol{r}', t)$$
$$+\mathrm{i}\frac{e^2}{m}(-\delta_{ij}(\boldsymbol{n}'\cdot\boldsymbol{n})+n'_in_j)\int\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r-r')}}{H(z)}\left(\frac{z^2}{r}+\frac{z}{r^2}\right)\left(\frac{z^2}{r'}+\frac{z}{r'^2}\right) \tag{4.2}$$

$$[E^i(\boldsymbol{r}, 0), B^j(\boldsymbol{r}', t)] = [E^i(\boldsymbol{r}, t), B^j(\boldsymbol{r}, 0)]$$
$$= -4\pi\mathrm{i}\epsilon^{ijs}\frac{\partial}{\partial r^s}\mathscr{D}_0(\boldsymbol{r}-\boldsymbol{r}', t)+\mathrm{i}\frac{e^2}{m}\epsilon^{pjs}\int_\Gamma\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{e}^{z(t-r-r')}}{H(z)}\left[\frac{z^2}{r}(\delta_{ip}-n_in_p)\right.$$
$$\left.+\left(\frac{z}{r^2}+\frac{1}{r^3}\right)(\delta_{ip}-3n_in_p)\right]n'_s\left(\frac{z^2}{r'}+\frac{z}{r'^2}\right) \tag{4.3}$$

where

$$\mathscr{D}_0(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3}\int\mathrm{d}_3k\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}}\frac{\sin kt}{k} \tag{4.4}$$

and ϵ^{ijk} denotes the fully antisymmetric tensor.

The first terms in these commutators correspond to free fields. Due to the second terms, the fields at two points $\boldsymbol{r}$ and $\boldsymbol{r}'$ do not commute when $t > r + r'$. This non-commutability of fields is related to the scattering process which makes these fields mutually dependent. Since it appears only when $t > r + r'$, we see that causality is also satisfied in the microscopic sense.

5. Final remarks

We have demonstrated that all requirements of relativistic causality are strictly fulfilled by the solutions of our model. This may be surprising, as the model is non-relativistic. One can verify, however, that the equations of motion (2.5) are equivalent to the set of Maxwell's equations with the point source. The only non-relativistic feature of our model is the approximation of the equation of motion for the charge. Due to the dipole approximation, the contribution to the Lorentz force from the magnetic field is neglected and the electric part of the force is position independent. The mass of the oscillating particle is assumed velocity independent.

This approximation only slightly affects the shape of emitted and scattered radiation. Propagation is governed by the free Maxwell equations, and is therefore relativistic.

Acknowledgments

The authors would like to thank Professors I Białynicki-Birula, F Rohrlich and P O Fröman and Drs J Ackerhalt and A Szymacha for interesting and useful discussions. This work was supported in part by NSF Grant No. GF 36217.

Appendix

Here we give some integrals useful in deriving the results of this paper.

$$\sum_{\mu} \int \mathrm{d}\Omega_{\boldsymbol{k}} e^{i}_{\boldsymbol{k}\mu} e^{j}_{\boldsymbol{k}\mu} \, \mathrm{e}^{\pm \mathrm{i}\boldsymbol{k}.\boldsymbol{r}} = 4\pi\left[(\delta_{ij} - n_i n_j)\frac{\sin kr}{kr} + (\delta_{ij} - 3n_i n_j)\left(\frac{\cos kr}{k^2 r^2} - \frac{\sin kr}{k^3 r^3}\right)\right] \tag{A.1}$$

$$\sum_{\mu} \int \mathrm{d}\Omega_{\boldsymbol{k}} (\boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{k}\mu})^{i} e^{j}_{\boldsymbol{k}\mu} \mathrm{e}^{\pm \mathrm{i}\boldsymbol{k}.\boldsymbol{r}} = \pm 4\pi \mathrm{i} \epsilon^{ijs} \frac{\partial}{\partial r^s} \frac{\sin kr}{kr}. \tag{A.2}$$

The following integrals are valid for Re $z > 0$:

$$\int_0^\infty \mathrm{d}x \frac{x \sin rx}{x^2 + z^2} = \frac{\pi}{2} \mathrm{e}^{-zr} \tag{A.3}$$

$$\int_0^\infty \mathrm{d}x \frac{\cos rx}{x^2 + z^2} = \frac{\pi}{2z} \mathrm{e}^{-zr} \tag{A.4}$$

$$\int_0^\infty \mathrm{d}x \frac{\sin xr}{x(x^2 + z^2)} = \frac{\pi}{2z^2}(1 - \mathrm{e}^{-zr}). \tag{A.5}$$

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